I. Review of the second derivative test in one variable.

Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a twice differentiable function on an open interval \( I \). Then the local extrema of \( f \) within \( I \) can be only at the critical points of \( f \), i.e. in the differentiable case only at points \( c \) where \( f'(c) = 0 \).

The proof of this statement is found in Salas-Hille-Etgen. An alternative, not quite rigorous, but intuitive argument is the following. We use the Taylor expansion of \( f \) around \( c \) up to first order:

\[
f(x) = f(c) + f'(c)(x - c) + o(|x - c|) \tag{1}
\]

where, as usual, \( o(|x - c|) \) means a function which vanishes at \( x = c \) faster than the function \( |x - c| \) (for example \((x - c)^2\)). Hence such a function is negligible compared to any nonzero linear function that vanishes at \( x = c \), at least for \( x \) close enough to \( c \). Hence if \( f'(c) \neq 0 \), then the trend of \( f(x) \) is determined by the sign of \( f'(c) \) (near \( x \approx c \)). If \( f'(c) > 0 \), then

\[
f(x) > f(c) \quad \text{for all} \quad x > c \quad \text{close enough to} \quad c
\]

and

\[
f(x) < f(c) \quad \text{for all} \quad x < c \quad \text{close enough to} \quad c
\]

If \( f'(c) < 0 \), then all inequalities are reversed. But in both cases \( f(x) \) has no local extremum at \( c \).

However, not all critical points are actually local extrema (e.g. \( f(x) = x^3 \) has critical point at \( c = 0 \), but it is neither local max. nor min.). Moreover, one would like to decide whether a local extremum is maximum or minimum.
The first derivative test is an appropriate tool, but it requires looking at the trend of the first derivative function $f'(x)$ around $c$, which may not be so easy.

A more direct test is the second derivative test: if $f'(c) = 0$, then $c$ is a local maximum if $f''(c) < 0$, it is a local minimum if $f''(c) > 0$. The test is inconclusive if $f''(c) = 0$.

Notice that the second derivative test requires slightly more about the function than the first derivative test; namely we need the existence of the second derivative (we need it only at the point $c$). Nevertheless, this test is much more appropriate to generalize for the multivariable case.

The rigorous proof of the second derivative test is found in Salas-Hille-Etgen. Here we give a different, not fully rigorous, but very intuitive argument.

The idea is that the expansion (1) is insufficient to decide about the trend of $f$ around $c$ if $f'(c) = 0$. But notice that (1) is actually a first-order Taylor expansion. Hence, we should expand it further!

So we use the Taylor expansion around $c$ up to second order:

$$f(x) = f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + o((x - c)^2)$$

where, as usual, $o((x - c)^2)$ means a function which vanishes at $x = c$ faster than the quadratic polynomial $(x - c)^2$ (for example $(x - c)^3$).

At the critical point we have $f'(c) = 0$, hence we have

$$f(x) - f(c) = \frac{1}{2}f''(c)(x - c)^2 + o((x - c)^2)$$

Now you can clearly see the second derivative test. Recall that an $o((x - c)^2)$ function is negligible compared to the quadratic polynomial $A(x - c)^2$ with any nonzero constant $A$, at least in a small neighborhood of $c$. Since $(x - c)^2 \geq 0$ we have the following for $x$ is close to $c$:

- if $f''(c) > 0$ then $f(x) - f(c) \geq 0$
- if $f''(c) < 0$ then $f(x) - f(c) \leq 0$

These are exactly the definitions of the local minimum and maximum, respectively. (In fact, we know even more: in the above inequalities there is equality only for $x = c$).

If $f''(c) = 0$, then the test is inconclusive, since then the $o((x - c)^2)$ term decides, which can have either sign. However, based upon this argument it is not hard to generalize the second partial derivative test by writing up higher order Taylor expansions and look for the first nonvanishing derivative. We do not go into these details.

II. Second order Taylor expansion in several variables.
Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a twice differentiable function on an open domain \( U \subset \mathbb{R}^n \) (i.e. all second partial derivatives exist).

The gradient \( \nabla f(c) \) at a point \( c \) gives the first order approximation to \( f \) around \( c \):
\[
f(x) = f(c) + \nabla f(c) \cdot (x - c) + o(x - c) \quad (3)
\]

In particular, there could be no local extremum unless \( \nabla f(c) = 0 \) since in that case the function increases with a positive rate in a certain direction (namely in the direction of \( \nabla f(c) \)).

So from now on we assume that \( \nabla f(c) = 0 \), these are the only candidates for local extrema (in case of differentiable function). How to decide whether it is maximum or minimum or none? Analogously to the one variable case, we would need a second order Taylor expansion. We know that the multivariable generalization of \( f'(c) \) is \( \nabla f(c) \) (compare formulas (1) and (3)), but what is the generalization of \( f''(c) \)?

We have all tools for the right guess. The second derivative, as its name suggests, must be the derivative of the first derivative, i.e. the derivative of \( \nabla f \). But \( \nabla f \) is an absolutely honest function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \). So its derivative is just the its Jacobian. Hence the “second derivative” of \( f \) is the Jacobian of \( \nabla f \)!

This is also called the Hessian of \( f \), and of course it is an \( n \) by \( n \) matrix:
\[
Hess_f := J_{\nabla f} = \begin{pmatrix}
\nabla (\frac{\partial}{\partial x_1} f) \\
\nabla (\frac{\partial}{\partial x_2} f) \\
\vdots \\
\nabla (\frac{\partial}{\partial x_n} f)
\end{pmatrix} = \begin{pmatrix}
f_{x_1x_1} & f_{x_2x_1} & \cdots & f_{xnx_1} \\
f_{x_1x_2} & f_{x_2x_2} & \cdots & f_{xnx_2} \\
\vdots & \vdots & \ddots & \vdots \\
f_{x_1xn} & f_{x_2xn} & \cdots & f_{xnxn}
\end{pmatrix}
\]

i.e. this is exactly the matrix of second partial derivatives. Sometimes the simpler notation \( f'' = Hess_f \) is also used, but be careful! If \( f \) is a multivariable function, then \( f'' \) is not a number but a matrix.

Here is a GOOD NEWS: by the symmetry of the second partials, \( f_{x_ix_j} = f_{x_jx_i} \), the Hessian is always a symmetric matrix.

Now we can easily guess the second order Taylor expansion in several variables, i.e. the generalization of (2). The only thing we have to generalize is the meaning of \( f''(c)(x - c)^2 \). Here \( f''(c) \) is an \( n \) by \( n \) matrix and \( (x - c) \) is an \( n \)-vector. The square of that vector makes sense (dot product) and it gives a number, so apparently \( f''(c)(x - c)^2 \) is a matrix times a number hence a matrix. But this can’t be the right thing here, since the left hand side of the expected analogue of (2) is clearly a number even in the multivariable case.

Hence we need a procedure which cooks up a number from a square matrix \( M \) and two vectors \( u, v \) (in fact here these two vectors will be the same, but the procedure is
There is not much choice, the right way to do that is that we let the matrix act on one of the vectors, then you get a vector and now take its dot product with the second vector: \( \mathbf{u}^t \cdot (M \mathbf{v}) \). This is called \textit{quadratic form}. It is easy to see that for symmetric matrix \( M \) it does not matter with vector you take for the first, since

\[
\mathbf{u}^t \cdot (M \mathbf{v}) = \mathbf{v}^t \cdot (M \mathbf{u})
\]

Applying it to our case, we have the \textbf{second order Taylor expansion in several variables}

\[
f(x) = f(c) + \nabla f(c) \cdot (x - c) + \frac{1}{2} (x - c)^t \cdot \left[ f''(c)(x - c) \right] + o((x - c)^2)
\]  

(4)

(\text{within the square bracket you have a matrix-vector multiplication, while the dot in front of the square bracket is a dot product.})

\textbf{III. Second derivative test in two variables.}

For simplicity, we restrict our attention to \( n = 2 \), but one can easily generalize the following argument.

Now we suppose that \( \nabla f(c) = 0 \). Is \( c \) local min, max, or none? Following (4), we look at

\[
f(x) = f(c) + \frac{1}{2} (x - c)^t \cdot \left[ f''(c)(x - c) \right] + o((x - c)^2)
\]  

(5)

hence the answer clearly depends on the matrix \( f''(c) \). For simplicity, we denote \( M := f''(c) \) (symmetric 2 by 2 matrix) and \( x - c = \mathbf{u} \). Remember, that \( M \) is fixed, but \( \mathbf{u} \) runs through all 2-vectors, and we can exclude the case \( \mathbf{u} = 0 \) from the investigation. What we have to decide is whether \( \mathbf{u}^t \cdot M \mathbf{u} \) is always positive, is always negative or none of these.

Since \( M \) is symmetric, we can find two real eigenvalues, \( \lambda_1, \lambda_2 \) and corresponding orthonormal eigenvectors \( \mathbf{v}_1, \mathbf{v}_2 \). By the spectral decomposition of \( M \), we have

\[
M = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^t + \lambda_2 \mathbf{v}_2 \mathbf{v}_2^t
\]

hence

\[
\mathbf{u}^t \cdot M \mathbf{u} = \lambda_1 (\mathbf{v}_1^t \cdot \mathbf{u})^2 + \lambda_2 (\mathbf{v}_2^t \cdot \mathbf{u})^2
\]

Since \( (\mathbf{v}_1^t \cdot \mathbf{u})^2 \) is always nonnegative and in fact

\[
\|\mathbf{u}\|^2 = (\mathbf{v}_1^t \cdot \mathbf{u})^2 + (\mathbf{v}_2^t \cdot \mathbf{u})^2 > 0
\]

we conclude that (\( \mathbf{u} = 0 \) excluded)

\begin{itemize}
  \item \( \mathbf{u}^t \cdot M \mathbf{u} \) is always positive if both eigenvalues are positive;
\end{itemize}
- $\mathbf{u}^T \cdot M \mathbf{u}$ is always negative if both eigenvalues are negative;
- $\mathbf{u}^T \cdot M \mathbf{u}$ is indefinite (i.e. it assumes both positive and negative values) if one eigenvalue is negative, the other is positive.
- $\mathbf{u}^T \cdot M \mathbf{u}$ can be zero even for nonzero $\mathbf{u}$ if one eigenvalue is zero.

These cases can easily be decided for a 2 by 2 symmetric matrix:

$$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

since the characteristic polynomial is

$$\lambda^2 - (A + C)\lambda + AC - B^2$$

Recall that the product of eigenvalues (roots) is $AC - B^2$ (which is by the way the determinant of $M$). Hence this number decides whether the two eigenvalues have the same or opposite signs or whether there is a zero eigenvalue. If the two eigenvalues have the same sign ($AC - B^2 > 0$) then clearly $A$ and $C$ must have the same sign. If this sign is positive, then the two eigenvalues are both positive, since their sum is $A + C$ (which is, by the way, the trace of $M$). If this sign is negative, then the eigenvalues are negative.

Now we can easily read off the second derivative test if you apply the analysis above to the matrix

$$M = f''(\mathbf{c}) = \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} f_{xx}(\mathbf{c}) & f_{xy}(\mathbf{c}) \\ f_{yx}(\mathbf{c}) & f_{yy}(\mathbf{c}) \end{pmatrix}$$

(here I used the notation $\mathbf{x} = (x, y)$ as usual). Following the notation of the book, we define $D := B^2 - AC$ (this is exactly the opposite of the quantity $AC - B^2$ we looked at above, I don’t know why the book uses this unnatural convention.)

I.e., if $D := B^2 - AC$ is positive, then the eigenvalues come with opposite signs; the second order term in the Taylor expansion (5) is indefinite, hence we have a saddle.

If $D := B^2 - AC$ is negative, then the eigenvalues have the same sign. If $A > 0$ (which is the same as $C > 0$) then this sign is positive, i.e. we have a local minimum. If $A < 0$, then we have local maximum.

Every other case is inconclusive (meaning that anything can happen).